Mass Spectra of Heavy Quarkonia and B_c Decay Constant for Static Scalar-Vector Interactions with Relativistic Kinematics

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Abstract

We reproduce masses of the self-conjugate and non-self-conjugate mesons in the context of the spinless Salpeter equation taking into account the relativistic kinematics and the quark spins. The hyperfine splittings for the 2S charmonium and 1S bottomonium are also calculated. Further, the pseudoscalar and vector decay constants of the B_c meson and the unperturbed radial wave function at the origin are also calculated. We have obtained a local equation with a complete relativistic corrections to a class of three attractive static interaction potentials of the general form $V(r) = -Ar^{-\beta} + \kappa r^{\beta} + V_0$, with $\beta = 1, 1/2,\ 3/4$ decomposed into scalar and vector parts in the form $V_V(r) = -Ar^{-\beta} + (1-\epsilon)\kappa r^{\beta}$ and $V_S(r) = \epsilon\kappa r^{\beta} + V_0$; where $0 \leq \epsilon \leq 1$. We have used the shifted large-N-expansion technique (SLNET) to solve the reduced equation for the scalar ($\epsilon = 1$), equal mixture of scalar-vector ($\epsilon = 1/2$), and vector ($\epsilon = 0$) confinement interaction kernels. The energy eigenvalues are carried out up to the third order approximation.

I. INTRODUCTION

The spinless Salpeter (SS) equation represents a standard approximation to the Bethe-Salpeter equation [1]. Upon elimination of any dependence on time-like variables in a suitable manner, the Bethe-Salpeter (BS) equation reduces to the Salpeter equation [2]. Neglecting, furthermore, any reference to the spin degrees of freedom and restricting to positive energy solutions, one arrives at the SS equation, which is the correct tool to deal with the bound-state spectrum of $q\overline{q}$, $Q\overline{Q}$ and $Q\overline{q}$ or $q\overline{Q}$ (where $q = u,\ d,\ s$ and $Q = c,\ b,\ t$) interacting

via some effective potential with no spin dependence, i.e., spin-averaged data (SAD). This equation was solved analytically and then numerically for its bound-state energies using different techniques by many authors [3-10].

We solved the SS equation [8-10] analytically and then numerically, for the first time, by using the shifted large-N-expansion technique (SLNET) for a wide class of static phenomenological and QCD-motivated potentials previously proposed for quarkonium [11-13]. The non-local SS equation was expanded formally to order of v^2/c^2 up to the first relativistic correction term. Thus, the resulting equation is a local Schrödinger-type equation for two constituent interacting particles of masses m_q and m_Q. On the other hand, it consists of heavy quarks and it can be reliably described by the use of the methods developed for the $c\overline{c}$ and the $b\overline{b}$ spectra.

The relativistic effects are not negligible in quarkonia, and we need to investigate them. The simplest relativistic effect is just kinematics. They are considered in the potential as nonstatic terms, including both spin-dependent and spin-independent parts. Therefore, the nonstatic terms in the potential, as in the Fermi-Breit approximation include a spin-spin hyperfine interaction, a tensor interaction, a spin-orbit interaction, and a spin-independent interaction [14-20]. There are also nonstatic terms in the potential which are independent of spin. It is difficult to isolate these terms from data, and furthermore, there is some controversy concerning their functional form [21]. In this situation, it is better to omit them altogether.

The discovery of the B_c (the lowest pseudoscalar $^1S_0$ state) was reported in 1998 by the collider Detector at Fermilab (CDF) collaboration in 1.8 TeV p-$\overline{p}$ collisions at the Fermilab [22]. The observed mass $M_{B_c} = 6.40 \pm 0.39 \pm 0.13\ GeV$ has inspired new theoretical interest in the problem [23-31]. Some preliminary estimates of the $c\overline{b}$ bound state masses have been made devoted to the description of the charmonium and bottomonium properties. More experimental data are expected to come in near future from new hadron colliders.

The spectroscopy of the $c\overline{b}$ system have already been widely studied various times over the past years in the framework of the heavy quarkonium theory [23]. The revised analysis

of the B_c spectroscopy has been performed in the framework of the potential approach and QCD sum rule [24, 27] . Kwong and Rosner [28] predicted the masses of the lowest $c\overline{b}$ vector (triplet) and pseudoscalar (singlet) states using an empirical mass formula and a logarithmic potential. Eichten and Quigg [22] gave a more comprehensive account of the energies and properties of the $c\overline{b}$ system that was based on the QCD-motivated potential of Buchmüller and Tye [32]. Gershtein *et al.* [29] also published a detailed account of the energies and decays of the $c\overline{b}$ system using a QCD sum-rule calculation. Baldicchi and Prosperi [27] have computed the $c\overline{b}$ spectrum based on an effective mass operator with full relativistic kinematics. They have also fitted the entire quarkonium spectrum. Fulcher *et al.* [25] extended the treatment of the spin-dependent potentials to the full radiative one-loop level and thus included effects of the running coupling constant in these potentials. They have also used the renormalization scheme developed by Gupta and Radford [32]. We applied the nonrelativistic form of the statistical model [30] and shifted large-N expansion technique (SLNET) [31] to calculate the spectroscopy, decay constant and other properties of the heavy mesons, including the $c\overline{b}$ system.

Recently, in 2002, the ALEPH collaboration has searched for the pseudoscalar bottomonium meson, the η_b in two-photon interactions at LEP2 with an integrated luminosity of 699 pb^{-1} collected at e^+e^- centre-of mass energies from 181 GeV to 209 GeV. One candidate event is found in the six-charged-particle final state and none in the four-charged-particle final state. The candidate η_b ($\eta_b \rightarrow K_S K^- \pi^+ \pi^- \pi^+$) has reconstructed invariant mass of $9.30 \pm 0.02 \pm 0.02$ GeV [33]. Theoretical estimates (from perturbative QCD and lattice nonrelativistic QCD of the mass splitting between $\eta_b(1S)$ and $\Upsilon(1S)$, $M(\Upsilon(1^3S_1)) = 9.460$ GeV, are reported (cf. [33] and references therein).

Further, in 2002, the Belle Collaboration [34] has observed a new pseudoscalar charmonium state, the $\eta_c(2S)$, in exclusive $B \longrightarrow K K_S K^- \pi^+$ decays. The measured mass of the $\eta_c(2S)$, $M(\eta_c(2S)) = 3654 \pm 14$ MeV. It is close to the $\eta_c(2S)$ mass observed by the same group in the experiment $e^+e^- \longrightarrow J/\psi\eta_c$ where $M(\eta_c(2S)) = 3622 \pm 12$ MeV was found [35]. It is giving rise to a small hyperfine splitting for the 2S state,

$\Delta_{\rm hfs}$(2S,exp)= $M(2^3{\rm S}_1) - M(2^1{\rm S}_0) = 32 \pm 14$ MeV [36]. Badalian and Bakker [37] calculated the hyperfine splitting for the 2S charmonium state in a recent work. Recksiegel and Sumino developed a new formalism [38] based on perturbative QCD to compute the hyperfine splittings of the bottomonium spectrum as well as the fine and hyperfine splittings of the charmonium spectrum [39].

The motivation of the present calculation is to extend the SLNET [8-10,31,40-42] to the treatment of the SS equation [10] to calculate the hyperfine splittings for the 2S charmonium and 1S bottomonium and also to reproduce the $c\overline{b}$ Salpeter binding masses below the continuum threshold $M(4S)$. We shall use a potential model that includes a running coupling constant effects in the central potential to give a simultaneous account of the properties of the $c\overline{b}$ system. We choose a class of three static central potentials [10-13,30,31,41,42] each with a strong coupling constant α_s to fit the spectroscopy of the existing and non-existing quarkonium systems. Since one would expect the average values of the momentum transfer in the various quark-antiquark states to be different, some variation in the values of the strong coupling constant and the normalization scale in the spin-dependent should be expected. We extend our previous work [31] for the solution of the SS equation to determine the binding masses of the $c\overline{c}$, $b\overline{b}$, and $c\overline{b}$ mesons by taking into account the spin-spin, spin-orbit and tensor interactions [14-21]. The spin effects are treated, in the framework of perturbation theory, as perturbation to the static potential and the treatments are based on the reduced Salpeter equation.

The outline of this paper is as following: In Section II, we first review briefly the analytic solution of the SS equation, we consider here the unequal mass case studied in our earlier works for a class of static potentials using SLNET. Section III is devoted for the class of three static potentials, which are decomposed into scalar and vector parts and also for their spin corrections. Pure vector and scalar potentials as well as their mixture are considered. Section IV is devoted for the pseudoscalar and vector leptonic constant of the B_c meson. We finally present an approximation to $c\overline{c}$, $b\overline{b}$, and $c\overline{b}$ mass spectra and decay constant of the B_c meson. We also calculate the the observed new charmonium η_c(2S) and searched bottomonium

η_c(1S) mesons and the hyperfine splittings of their states. Finally, our conclusions are given in Section V. Appendix A contains some definitions as well as the formulas necessary to carry out the above mentioned computations.

II. WAVE EQUATION

The relativistic wave Salpeter equation [8-10] is constructed by considering the kinetic energies of the constituents and the interaction potential. We assume that the full Hamiltonian H governing the dynamics of the quantum-mechanical system under consideration can be split up into a free Hamiltonian H_0 and an interaction potential $V(r)$. For the case of two particles with unequal masses m_q and m_Q, interacting via a spherically symmetric potential $V(r)$, the spatial coordinate representation of the SS equation in the center-of-momentum system for the two-body system reads off

$$\left[\sum_{i=q,Q}\sqrt{-\Delta_N+m_i^2}+V(r)-M_{n,l}(q\overline{Q})\right]\Psi\left(\mathbf{r}\right)=0, \tag{1}$$

with the potential $V(r)$ contains, in addition to the static interaction $V_{static}(r)$ the total spin-dependent potential $V_{SD}(r)$ [15-18,20,21]. The kinetic terms involving the operation $\sqrt{-\Delta_N+m_i^2}$ are nonlocal operators and can be solved through a direct expansion of the square root-operator [5,10,17]. For heavy quarks, the kinetic energy operators in Eq. (1) can be approximated by the standard expansion in the inverse powers of quark masses to obtain the usual nonrelativistic Schrödinger Hamiltonian [5,10]

$$\sum_{i=q,Q}\sqrt{-\Delta_N+m_i^2}=m_q+m_Q-\frac{\Delta_N}{2\mu}-\frac{\Delta_N^2}{8\eta^3}-\cdots, \tag{2}$$

where $\mu=\frac{m_qm_Q}{m_q+m_Q}$ denotes the reduced mass and $\eta=\mu\left(\frac{m_qm_Q}{m_qm_Q-3\mu^2}\right)^{1/3}$ is a useful mass parameter. The radial part of Eq. (1), in the N-dimensional space, is expanded in powers of v^2/c^2 up to one relativistic correction term as [5,8-10,16]

$$\left\{-\frac{\Delta_N}{2\mu}-\frac{\Delta_N^2}{8\eta^3}+V(r)\right\}R_{n,l}(r)=E_{n,l}R_{n,l}(r), \tag{3}$$

where $E_{n,l} = M_{n,l}(q\overline{Q}) - m_q - m_Q$ stands for the Salpeter binding energy and $\Delta_N = \nabla_N^2$. [1] This SS-type equation retains its relativistic kinematics and is suitable for describing the spin-averaged spectrum of two bound quarks of masses m_q and m_Q and total binding meson mass eigenstate $M_{n,l}(q\overline{Q})$. Furthermore, in order to obtain a Schrödinger-like equation, the perturbed term in Eq. (3) is treated using the reduced Schrödinger equation [44]

$$p^4 = 4\mu^2 \left[E_{n,l} - V(r)\right]^2, \tag{4}$$

where p is the center of mass momentum of the $q\overline{Q}$ system. Consequently one would reduce Eq. (3) to the Schrödinger-type form [17]

$$\left\{-\frac{\Delta_N}{2\mu} - \frac{\mu^2}{2\eta^3}\left[E_{n,l}^2 + V^2(r) - 2E_{n,l}V(r)\right] + V(r)\right\} R_{n,l}(\mathbf{r}) = E_{n,l}R_{n,l}(\mathbf{r}). \tag{5}$$

After employing the following transformation

$$R_{n,l}(r) = \frac{u(r)}{r^{(N-1)/2}}, \tag{6}$$

we may rewrite Eq.(5) simply as

$$\left[-\frac{1}{2\mu}\frac{d^2}{dr^2} + \frac{\left[\overline{k} - (1-a)\right]\left[\overline{k} - (3-a)\right]}{8\mu r^2} + W_{n,l}(r) - \frac{W_{n,l}(r)^2}{2m'}\right] u_{n,l}(r) = 0, \tag{7}$$

with

$$W_{n,l}(r) = V_{eff}(r) - E_{n,l}, \tag{8}$$

and the effective mass parameter

$$m' = \eta^3/\mu^2 = (m_q m_Q \mu)/(m_q m_Q - 3\mu^2). \tag{9}$$

The perturbation term, $W_{n,l}(r)^2$, in (7) is significant only where it is small (i.e., $W_{n,l}(r)/m' \ll 1$). This condition is verified by the confining potentials used to describe heavy−quark

[1] This approximation is correct to $O(v^2/c^2)$. The Δ_N^2 term in (3) should be properly treated as a perturbation by using trial wavefunctions [43].

systems except near the color−Coulomb singularity at the origin, and for $r \to \infty$. However, it is always satisfied on the average as stated by Durand et al. [17]. We now proceed to solve Eq. (7) with Eq. (8). In the SLNET [8-10,30,40-42], it is convenient to shift the origin of coordinates to $r = r_0$ (or $y = 0$) by defining

$$y = \bar{k}^{1/2}(r - r_0)/r_0. \tag{10}$$

Expansions about this point in powers of y and $\overline{k}$ yield

$$\frac{1}{r^2} = \frac{1}{r_0^2}\sum_{j=0}^{\infty}(-y)^j\frac{(1+j)}{\overline{k}^{-j/2}}, \tag{11}$$

$$V(y(r_0)) \ = \frac{1}{Q}\sum_{j=0}^{\infty}\left(\frac{d^jV(r_0)}{dr_0^j}\right)\frac{(r_0y)^j}{j!}\overline{k}^{(4-j)/2}, \tag{12}$$

and also

$$E_{n,l} \ = \frac{1}{Q}\sum_{j=0}^{\infty}\overline{k}^{(2-j)}E_j. \tag{13}$$

Further, by substituting Eqs. (8) through (13) into Eq. (7), one gets

$$\left\{-\frac{1}{4\mu}\frac{d^2}{dy^2} + \left[\frac{\overline{k}}{16\mu} - \frac{(2-a)}{8\mu} + \frac{(1-a)(3-a)}{16\mu\overline{k}}\right]\times\sum_{j=0}^{\infty}(-y)^j\frac{(1+j)}{\overline{k}^{j/2}}\right.$$

$$+\frac{r_0^2}{Q}\sum_{j=0}^{\infty}\left(\frac{d^jV(r_0)}{dr_0^j}\right)\frac{(r_0y)^j}{j!}\overline{k}^{(2-j)/2} - \frac{r_0^2\,\overline{k}}{m'Q}\left[\sum_{j=0}^{\infty}\left(\frac{d^jV(r_0)}{dr_0^j}\right)\frac{(r_0y)^j}{j!}\overline{k}^{-j/2}\right]^2$$

$$\left. + \frac{2r_0^2}{m'Q}\sum_{j=0}^{\infty}\overline{k}^{(1-j)}E_j\times\sum_{j=0}^{\infty}\left(\frac{d^jV(r_0)}{dr_0^j}\right)\frac{(r_0y)^j}{j!}\overline{k}^{-j/2}\right\}\varphi_{n_r}(y) = \mathcal{E}_{n_r}\varphi_{n_r}(y), \tag{14}$$

with the eigenvalues

$$\mathcal{E}_{n_r} = \frac{r_0^2}{Q}\left\{\bar{k}\left(E_0 + \frac{E_0^2}{m'}\right) + \left(E_1 + \frac{2E_0E_1}{m'}\right) + \left(E_2 + \frac{2E_0E_2}{m'} + \frac{E_1^2}{m'}\right)\frac{1}{\bar{k}}\right.$$

$$\left. + \left(E_3 + \frac{2E_0E_3}{m'} + \frac{2E_1E_2}{m'}\right)\frac{1}{\bar{k}^2} + \cdots\right\}. \tag{15}$$

The parameter Q is an arbitrary scale, but is to be set equal $\overline{k}^2$ at the end of our calculations. Thus comparing Eqs. (14) and (15) with its counterpart Schrödinger−like equation for the one−dimensional anharmonic oscillator problem [40], we calculate all the relevant quantities δ's and ε's [31]. The final analytic expression in the $1/\bar{k}$ expansion of the energy eigenvalues appropriate to the SS particle is

$$\mathcal{E}_{n_r} = \bar{k}\left[\frac{1}{16\mu} + \frac{r_0^2 V(r_0)}{Q} - \frac{r_0^2 V(r_0)^2}{m'Q} + \frac{2r_0^2 E_0 V(r_0)}{m'Q}\right]$$

$$+\left[(1+2n_r)\frac{\omega}{2} - \frac{(2-a)}{8\mu}\right] + \frac{1}{\bar{k}}\left[\frac{2r_0^2 E_2 V(r_0)}{m'Q} + \beta^{(1)}\right]$$

$$+\frac{1}{\bar{k}^2}\left[\frac{2r_0^2 E_3 V(r_0)}{m'Q} + \beta^{(2)}\right] + O\left[\frac{1}{\bar{k}^3}\right], \tag{16}$$

with n_r is to be set equal to $0, 1, 2, \cdots$. The quantities $\beta^{(1)}$ and $\beta^{(2)}$ appearing in the correction to the leading order of the energy expression are displayed in [31].

Comparing the terms of Eq. (15) with their counterparts in Eq. (16) and equating terms of same order in $\overline{k}$, one gets the leading contribution to the binding energy

$$E_0 = V(r_0) + \frac{m'}{2}\left[\sqrt{1 + \frac{Q}{4r_0^2 m_\mu m'^2}} - 1\right], \tag{17}$$

where m_μ=μ/m'. Here r_0 is chosen to minimize E_0, that is,

$$\frac{dE_0}{dr_0} = 0 \ ; \quad \frac{d^2E_0}{dr_0^2} > 0, \tag{18}$$

and it satisfies

$$r_0^3 V'(r_0)\left(\frac{m'^2}{4} + \frac{Q}{16r_0^2 m_\mu}\right)^{1/2} = \frac{Q}{16m_\mu}. \tag{19}$$

To solve for the shifting parameter a, the next contribution to the energy eigenvalue, in Eq. (13), is chosen to vanish (i.e., $E_1 = 0$) which gives

$$a = 2 - 4\mu(1+2n_r)\omega, \tag{20}$$

with ω defined as

$$\omega = \frac{1}{4\mu}\Big[3 + r_0 V''(r_0)/V'(r_0) - 16 r_0^4 m_\mu V'(r_0)^2/Q\Big]^{1/2}. \tag{21}$$

The scaling parameter Q, in Eq. (19), is simply written as

$$Q = 8m_\mu\Big[\, r_0^2 V'(r_0)\Big]^2 (1+\xi), \tag{22}$$

with

$$\xi = \sqrt{1 + \left(\frac{m'}{r_0 V'(r_0)}\right)^2}. \tag{23}$$

Therefore, with the help of relations (20) through (23) together with $Q = \overline{k}^2$, we obtain the following formula

$$1 + 2l + 4\mu(2n_r + 1)\omega = 2r_0^2 V'(r_0)\left(2m_\mu + 2m_\mu\xi\right)^{1/2}, \tag{24}$$

which is an explicit equation in r_0. Once r_0 is determined via (24), it becomes easy and straightforward to obtain E_0 via Eq. (17), E_2 and E_3 via solving Eqs. (15) -(16). Thus, the general expression for the binding energy eigenvalue takes

$$E_{n,l} = E_0 + \frac{1}{r_0^2\left(1 - \frac{2W_{n,l}(r_0)}{m'}\right)}\left[\beta^{(1)} + \frac{\beta^{(2)}}{\bar{k}} + O\left(\frac{1}{\bar{k}^2}\right)\right], \tag{25}$$

which works well and is convergent as the value of l increases. Finally, the binding meson mass eigenstate for the quarkonium families is

$$M_{n,l}(q\overline{Q}) = 2E_{n,l} + m_q + m_Q. \tag{26}$$

where m_1 and m_2 are constituent quark masses.

III. HEAVY QUARKONIUM AND B_C MESON MASS SPECTRA

To describe the spin-dependent relativistic corrections to the potential, we have used a similar approach derived by Olsson *et al.* [14] for positron. We choose the potential in Eq. (1) as [18,20,44]

$$V(r) = V_{static}(r) + V_{SD}(r) + V_{SI}(r), \tag{27}$$

with spin-dependent and spin-independent perturbation terms are given in Refs. [20,21,43,44]. Further, the static potential [7-13,30,31,41,42] takes the general form

$$V_{static}(r) = -\frac{A}{r^{\beta}} + \kappa r^{\beta} + V_0;\ \beta = 1, 1/2,\ 3/4, \tag{28}$$

where $A > 0,\ \kappa > 0$ and V_0 may be of either sign. The first potential we consider here is the Cornell [13] potential which is one of the earliest QCD-motivated potentials having the form

$$V_C(r) = -\frac{A}{r} + \kappa r + V_0, \tag{29}$$

where $A = 4\alpha_s/3$, the Coulomb-type piece characterizes the short-range gluon exchange, κ is a confinement constant, and the constant V_0 is for spin-independent interactions not included explicitly in the $r-$ dependent part of the potential. It is related to the slope κ of the linear potential by [15]

$$V_0 = -2\sqrt{\kappa}\exp(-\gamma_E + \frac{1}{2}), \tag{30}$$

where $\gamma_E = 0.577215\cdots$ is the Euler-Mascheroni constant. The second potential is that of Song and Lin [12] and is given by

$$V_{S-L}(r) = -\frac{A}{r^{1/2}} + \kappa r^{1/2} + V_0. \tag{31}$$

The third potential is an intermediate case between the last two mentioned potentials and is called Turin potential [11]

$$V_T(r) = -\frac{A}{r^{3/4}} + \kappa r^{3/4} + V_0. \tag{32}$$

The class of static potentials in Eq. (28) must satisfy the following conditions

$$\frac{dV}{dr} > 0,\ \frac{d^2V}{dr^2} \leq 0. \tag{33}$$

The fine and hyperfine splittings are computed by using the Breit-Fermi interaction with the assumptionp [16,18]

$$V_S(r) = \epsilon\kappa r^\beta + V_0, \tag{34}$$

$$V_V(r) = V_{static}(r) - V_S(r), \tag{35}$$

where ϵ is some mixing parameter. The vector term incorporates the expected short-distance behaviour from single-gluon exchange. We have also included a multiple of the long-range interaction in $V_V(r)$ to see if we can determine the vector-scalar nature of the confining interaction. Here we investigate the cases of pure scalar confinement ($\epsilon = 1$), equal mixture of scalar-vector couplings ($\epsilon = 1/2$) and a pure vector case ($\epsilon = 0$). The spin dependent correction to the nonrelativistic Hamiltonian, which is responsible for the fine splittings, is also modelled on the Breit-Fermi Hamiltonian [22,29]. It can be decomposed into a part, which is antisymmetric with respect to the spins of the constituents $V_A(r)$, and apart symmetric in these spins. The symmetric part can be decomposed into a spin-orbit interaction $V_{LS}(r)$, a spin-spin interaction $V_{SS}(r)$, and a tensor part $V_T(r)$. Therefore, it is given by [15-17,20,21,44]

$$V_{SD}(r) = V_A + V_S = \frac{1}{4}\left[\frac{1}{m_q^2} - \frac{1}{m_Q^2}\right]\left[\frac{V_V'(r) - V_S'(r)}{r}\right]\mathbf{L}\cdot\mathbf{S}_-$$

$$+\frac{\mathbf{L}\cdot\mathbf{S}}{m_q m_Q}\frac{V_V'(r)}{r} + \frac{1}{2}\left[\frac{\mathbf{L}\cdot\mathbf{S}_1}{m_q^2} + \frac{\mathbf{L}\cdot\mathbf{S}_2}{m_Q^2}\right]\left[\frac{V_V'(r) - V_S'(r)}{r}\right]$$

$$+\frac{2}{3}\frac{\mathbf{S}_1\cdot\mathbf{S}_2}{m_q m_Q}\left[\nabla^2 V_V(r)\right] + \frac{S_{12}}{m_q m_Q}\left[-V_V''(r) + \frac{V_V'(r)}{r}\right], \tag{36}$$

where $\mathbf{S}_1$ and $\mathbf{S}_2$ are the quark spins, $\mathbf{S}_- = \mathbf{S}_1 - \mathbf{S}_2$, $\mathbf{L} = \mathbf{x}\times\mathbf{p}$ is the relative orbital angular momentum, and $S_{12} = T - (\mathbf{S}_1\cdot\mathbf{S}_2)/3$ where $T = (\mathbf{S}_1\cdot\widehat{\mathbf{r}})(\mathbf{S}_2\cdot\widehat{\mathbf{r}})$ is the tensor operator with the versor $\widehat{\mathbf{r}} = \mathbf{r}/r$. The spin dependent correction (36), which is responsible for the hyperfine splitting of the mass levels, in the short-range is generally used in the form for S-wave ($L = 0$) (cf. e.g., [32,43,44])

$$V_{HFS}(r) = \frac{2}{3}(\mathbf{S}_1\cdot\mathbf{S}_2)\nabla^2\left[-\frac{4\alpha_s}{3r^\beta}\right], \tag{37}$$

but the one responsible for the fine splittings is used for P- and D-waves ($L \neq 0$) it is given by

$$V_{FS}(r) = \frac{1}{m_q m_Q}\left\{\frac{\mathbf{L}\cdot\mathbf{S}}{r}\left[\left(1+\frac{1}{4}\frac{m_q^2+m_Q^2}{m_q m_Q}\right)V_V'(r) - \frac{1}{4}\frac{m_q^2+m_Q^2}{m_q m_Q}V_S'(r)\right]\right.$$

$$\left. +\frac{2}{3}(\mathbf{S}_1\cdot\mathbf{S}_2)\nabla^2\left[\kappa(1-\epsilon)r^\beta\right] + \left[T - \frac{1}{3}\left(\mathbf{S}_1\cdot\mathbf{S}_2\right)\right]\left[V_V''(r) + \frac{V_V'(r)}{r}\right]\right\}, \tag{38}$$

where the matrix element can be evaluated in terms of the expectation values [22] $\langle\mathbf{L}\cdot\mathbf{S}_1\rangle = \langle\mathbf{L}\cdot\mathbf{S}_2\rangle = \frac{1}{2}\langle\mathbf{L}\cdot\mathbf{S}\rangle$. Hence, Eq. (38) is the complete spin-dependent potential in QCD through order m^2. For bound state constituents of spin $S_1 = S_2 = 1/2$, the scalar product of their spins $\mathbf{S}_1\cdot\mathbf{S}_2$ and $\mathbf{L}\cdot\mathbf{S}$ are to be found in the Appendix A. The appearance of a Coulomb-like contribution $\sim 1/r$ in the vector part of the potential causes some problems due to the relation $\nabla^2(1/r) = -4\pi\delta^{(3)}(x)$, in the spin-spin interaction (37) involves a delta function of the S-wave ($L = 0$). Thus, for Cornell potential, the hyperfine splitting potential (37) gives

$$V_{eff}(r) = -\frac{A}{r} + \kappa r + \frac{32\pi\alpha_s}{9m_q m_Q}\delta^{(3)}(\mathbf{r})(\mathbf{S}_1\cdot\mathbf{S}_2) + V_0;\ \text{where } \beta = 1. \tag{39}$$

Therefore for the energy of spin-spin interaction we have approximately:

$$E_{ss} = \frac{1}{2M_{n,0}}\Delta M_{ss}^2\left\langle\mathbf{S}_1\cdot\mathbf{S}_2\right\rangle, \tag{40}$$

where $M_{n,0}$ is given in Eq. (26) and the singlet-triplet mass squared difference [15]

$$\Delta M_{ss}^2 = M_{S=1}^2 - M_{S=0}^2 \simeq \frac{32}{9}\alpha_s\kappa, \tag{41}$$

for light $q\overline{q}$ systems (in the instantaneous-limit approximation) [15], and

$$\Delta M_{ss}^2 = M_{S=1}^2 - M_{S=0}^2 \simeq \frac{256}{3\pi^2}\alpha_s\kappa, \tag{42}$$

for heavy quarkonia (hydrogen-like trial functions) [15]. All these predictions for the mass-squared difference are independent of the mass of the particles which constitute the bound state. Further, for the Song-Lin and Turin potentials, it also give

$$V_{eff}(r) = -\frac{A}{r^{\beta}} + \kappa r^{\beta} + \frac{8\beta(1-\beta)\pi\alpha_s}{9m_q m_Q r^2} r^{-\beta} \mathbf{S}_1 \cdot \mathbf{S}_2 + V_0; \text{ where } \beta = 1/2, 3/4. \tag{43}$$

On the other hand, the Eq. (38), for P, $D, \cdots$ waves ($L \neq 0$) case, gives

$$V_{eff}(r) = V_{static}(r) + g(r)\left[F_{LS_-}\left(\mathbf{L}\cdot\mathbf{S}_-\right) + F_{LS}\left(\mathbf{L}\cdot\mathbf{S}\right) + F_{SS}\left(\mathbf{S}_1\cdot\mathbf{S}_2\right) + F_T T\right], \tag{44}$$

with a given set of spin-dependent quantities

$$F_{LS_-} = \left[\frac{1}{4}\frac{m_Q^2 - m_q^2}{m_q m_Q}\left[Ar^{-\beta} + (1-\epsilon)\,\kappa r^{\beta}\right]\right], \tag{45}$$

$$F_{LS} = \left[\left(1 + \frac{1}{4}\frac{m_q^2 + m_Q^2}{m_q m_Q}\right)\left[Ar^{-\beta} + (1-2\epsilon)\,\kappa r^{\beta}\right] + \epsilon\kappa r^{\beta}\right], \tag{46}$$

$$F_{SS} = \left[-\frac{(2+\beta)}{3}Ar^{-\beta} + \beta\,(1-\epsilon)\,\kappa r^{\beta}\right], \tag{47}$$

and

$$F_T = \left[(2+\beta)Ar^{-\beta} + (2-\beta)\,(1-\epsilon)\,\kappa r^{\beta}\right], \tag{48}$$

where $g(r) = \frac{\beta}{m_q m_Q r^2}$ is a necessary coupling function. The spin-independent corrections in Eq. (27) are explicitly given by Ref. [44] which are not treated in our present work.

A. Singlet states

For parastates ($L = J$) or ($S = 0$) case, we have parity $P = (-1)^{J+1}$ and charge conjugation $C = (-1)^L$. Thus, the potential (44) can be rewritten as

$$V_{eff}(r) = V_{static}(r) - \frac{1}{4}\left(3F_{SS} + F_T\right) + \sqrt{\frac{1}{10}(2L+3)(2L-1)}F_{LS_-}, \tag{49}$$

which can be substituted in Eq. (8) and also by setting $\overline{k} = N + 2J - a$ therein. Further, Eqs. (39) and (43) give

$$V_{eff}(r) = -\frac{A}{r} + \kappa r - \frac{8\pi\alpha_s}{3m_q m_Q}\delta^{(3)}(\mathbf{r}) + V_0, \tag{50}$$

and

$$V_{eff}(r) = -\frac{A}{r^{\beta}} + \kappa r^{\beta} - \frac{2\beta(1-\beta)\pi\alpha_s}{3m_q m_Q r^2} r^{-\beta} + V_0; \text{ where } \beta = 1/2, 3/4, \tag{51}$$

respectively, which generate singlet states with opposite quark and antiquark spins of the signature n^1S_0. Furthermore, Eq. (49) can be rewritten simply as

$$V_{J=L}(r) = g(r) \left\{ \frac{1}{4} \frac{m_Q^2 - m_q^2}{m_q m_Q} \sqrt{\frac{1}{10}(2L+3)(2L-1)} \left[A r^{-\beta} + (1-\epsilon)\kappa r^{\beta} \right] \right.$$

$$\left. - \frac{1}{2}(1+\beta)(1-\epsilon)\kappa r^{\beta} \right\} + V_{static}(r) \tag{52}$$

which generates states of the signatures n^1P_1, n^1D_2, n^1F_3, $n^1G_4, \cdots$.

B. Triplet states

For triplet ($S = 1$) case, we have the known inequality $|L - S| \leq J \leq L + S$ that gives $J = L$ and $L \pm 1$:

1. States $J = L$

Here, the parity $P = (-1)^{J+1}$and the charge conjugation $C = (-1)^{L+1}$. The potential in Eq. (44) takes the following simple form

$$V_{eff}(r) = V_{static}(r) + \frac{1}{4}\left(F_{SS} + F_T - 4F_{LS}\right) + \sqrt{\frac{1}{10}(2L+3)(2L-1)} F_{LS_-}, \tag{53}$$

which can be substituted in (8) together with $\overline{k} = N + 2J - a$ therein. Further, the potential (53) reads

$$V_{J=L}(r) = -\frac{g(r)}{2} \left\{ \left[\left(\frac{4-\beta}{3} + \frac{1}{2}\frac{m_q^2 + m_Q^2}{m_q m_Q} \right) A r^{-\beta} + \left(1 + \frac{1}{2}\frac{m_q^2 + m_Q^2}{m_q m_Q} \right)(1-2\epsilon)\kappa r^{\beta} + \epsilon\kappa r^{\beta} \right] \right.$$

$$\left. - \frac{1}{2}\frac{m_Q^2 - m_q^2}{m_q m_Q} \sqrt{\frac{1}{10}(2L+3)(2L-1)} \left[A r^{-\beta} + (1-\epsilon)\kappa r^{\beta} \right] \right\} + V_{static}, \tag{54}$$

which generates states like $n^3P_1,\ n^3D_2,\ n^3F_3,\ n^3G_4, \cdots$.

2. States $J = L \pm 1$

We have the parity $P = (-1)^J$ and the charge conjugation $C = (-1)^{L+1}$. The eigenfunction is a superposition of two components with orbital momentum $L = J+1$ and $L = J-1$ which have equal space parity

$$\psi_{S,J}(r) = \widehat{u}_{J-1}(r) Y^m_{J-1,1,J}(\theta,\varphi) + \widehat{u}_{J+1}(r) Y^m_{J+1,1,J}(\theta,\varphi). \tag{55}$$

The action of the tensor operator, T, on the two components of the wavefunction in Eq. (55) is

$$T u_{J\pm1} Y^m_{J\pm1,1,J}(\widehat{\mathbf{r}}) = \mp \frac{1}{4(2J+1)} u_{J\pm1} Y^m_{J\pm1,1,J}(\widehat{\mathbf{r}}) + \frac{1}{2}\frac{\sqrt{J(J+1)}}{2J+1} u_{J\mp1} Y^m_{J\mp1,1,J}(\widehat{\mathbf{r}}). \tag{56}$$

Therefore, a set of equations are obtained

$$\left\{ -\frac{1}{2\mu}\frac{d^2}{dr^2} + \frac{\left[\overline{k}-(1-a)\right]\left[\overline{k}-(3-a)\right]}{8\mu r^2} + V_{static}(r) + \frac{1}{4}\left(F_{SS} - \frac{F_T}{(2J+1)}\right) - E_{n,J+1} \right.$$

$$\left. -(J+2)\,F_{LS} - \frac{1}{2m'}\left[V_{static}(r) - (J+2)\,F_{LS} + \frac{1}{4}\left(F_{SS} - \frac{F_T}{(2J+1)}\right) - E_{n,J+1}\right]^2 \right\}$$

$$\times \widehat{u}_{n,J+1}(r) - \frac{\sqrt{J(J+1)}}{2\,(2J+1)} F_T \widehat{u}_{n,J-1}(r) = 0, \tag{57}$$

and

$$\left\{ -\frac{1}{2\mu}\frac{d^2}{dr^2} + \frac{\left[\overline{k}-(1-a)\right]\left[\overline{k}-(3-a)\right]}{8\mu r^2} + V_{static}(r) + \frac{1}{4}\left(F_{SS} + \frac{F_T}{(2J+1)}\right) - E_{n,J-1} \right.$$

$$\left. +(J-1)\,F_{LS} - \frac{1}{2m'}\left[V_{static}(r) + (J-1)\,F_{LS} + \frac{1}{4}\left(F_{SS} + \frac{F_T}{(2J+1)}\right) - E_{n,J-1}\right]^2 \right\}$$

$$\times \widehat{u}_{n,J-1}(r) - \frac{\sqrt{J(J+1)}}{2\,(2J+1)} F_T \widehat{u}_{n,J+1}(r) = 0, \tag{58}$$

where $\overline{k} = N + 2J + 2 - a$. Therefore, Eqs. (57) and (58) describe states such as $n^3P_2,\ n^3D_3,\ n^3F_2,\ n^3H_4, n^3P_0,\ n^3D_1, \cdots$. Here we may consider numerically the system

obtained and separate equations by dropping out the mixed terms to see their effect on the spectrum of the masses. Consequently one can rewrite (57) and (58) in the following simplest forms

$$V_{J=L-1}(r) = -g(r)\left\{(L+1)\left[\left(1+\frac{1}{4}\frac{m_q^2+m_Q^2}{m_q m_Q}\right)\left[Ar^{-\beta}+(1-2\epsilon)\,\kappa r^{\beta}\right]+\epsilon\kappa r^{\beta}\right]\right.$$

$$+\frac{1}{4}\frac{1}{(2L-1)}\left[(2+\beta)Ar^{-\beta}+(2-\beta)\,(1-\epsilon)\,\kappa r^{\beta}\right]$$

$$\left.+\frac{1}{4}\left[\frac{(2+\beta)}{3}Ar^{-\beta}-\beta\,(1-\epsilon)\,\kappa r^{\beta}\right]\right\}+V_{static}(r), \tag{59}$$

for states $n^3P_0,\ n^3D_1,\ n^3F_2,\ n^3H_4,\cdots$ and

$$V_{J=L+1}(r) = g(r)\left\{\left[\left(1+\frac{1}{4}\frac{m_q^2+m_Q^2}{m_q m_Q}\right)\left[Ar^{-\beta}+(1-2\epsilon)\,\kappa r^{\beta}\right]+\epsilon\kappa r^{\beta}\right]L\right.$$

$$+\frac{1}{4}\frac{1}{(2L+3)}\left[(2+\beta)Ar^{-\beta}+(2-\beta)\,(1-\epsilon)\,\kappa r^{\beta}\right]$$

$$\left.-\frac{1}{4}\left[\frac{(2+\beta)}{3}Ar^{-\beta}-\beta\,(1-\epsilon)\,\kappa r^{\beta}\right]\right\}+V_{static}(r), \tag{60}$$

for states $n^3P_2,\ n^3D_3,\cdots$. Further, for triplet S-wave, we have

$$V_{eff}(r) = -\frac{A}{r}+\kappa r+\frac{8\pi\alpha_s}{9m_q m_Q}\delta^{(3)}(\mathbf{r})+V_0, \tag{61}$$

and

$$V_{eff}(r) = -\frac{A}{r^{\beta}}+\kappa r^{\beta}+\frac{2\beta(1-\beta)\pi\alpha_s}{9m_q m_Q r^2}r^{-\beta}+V_0;\ \text{where}\ \beta=1/2,3/4, \tag{62}$$

which describe states such as n^3S_1.

3. State $J=0$

Equations (59) and (60) degenerate into a single equation with an effective potential

$$V_{eff}(r) = V_{static}(r)+\frac{1}{4}(F_{SS}-F_T-8F_{LS}), \tag{63}$$

and also by setting $\overline{k} = N + 2 - a$ therein. Further, Eq. (63) becomes

$$V_{J=0}(r) = -\left\{1 + g(r)\left[2 + \frac{1}{2}\frac{m_q^2 + m_Q^2}{m_q m_Q} + \frac{2+\beta}{3}\right]\right\} Ar^{-\beta}$$

$$+\left\{1 - \frac{g(r)}{2}\left[(5-\beta)(1-\epsilon) + \left(\frac{m_q^2 + m_Q^2}{m_q m_Q}\right)(1-2\epsilon)\right]\right\} \kappa r^{\beta} + V_0. \tag{64}$$

which only describes states such as n^3P_0.

IV. PSEUDOSCALAR AND VECTOR DECAY CONSTANTS OF THE B_C MESON

The significant contribution to the B_c total decay rate comes from the annihilation of the c quark and $\overline{b}$ antiquark into the vector boson W^+ which decays into a lepton and a neutrino or a quark-antiquark pair.

The nonrelativistic expression for the decay constants is given by [45-47]

$$f_P^{NR} = f_V^{NR} = \sqrt{\frac{12}{M_{P,V}(q\overline{Q})}}\,|\Psi_{P,V}(0)| \tag{65}$$

where $\Psi_{P,V}(0)$ is the meson wave function at the origin $r = 0$. The f_P and f_V, P corresponds to the pseudoscalar B_c and V to to the vector B_c^* mesons and $M_{P,V}(q\overline{Q})$ are the masses of the B_c and B_c^* mesons.

V. RESULTS AND CONCLUSION

We have solved the spinless Salpeter equation using SLNET and also extended our earlier formalism for the SAD spectra [31] by introducing the spin corrections. Furthermore, we have obtained a unified description of the self-conjugate meson spectroscopy with a phenomenological and a QCD-motivated potential model. This model has also extended to comprise various cases of pure scalar confinement ($\epsilon = 1$), scalar-vector couplings ($\epsilon = 1/2$) and the vector confinement ($\epsilon = 0$) interactions. It was motivated by our desire to construct

an analytic expression for the quarkonium mass spectra of different spins and confining interactions that could have combined both heavy and light quarkonia.

The parameter fits used in this work together with the quark masses are shown in Table I. They are taken to be same as those considered in our earlier works [10] for each static attractive potential. We do not view our present result as a totally successful model of quarkonia interactions even though the level structure is good and the transitions are also well accounted for. It seems likely that in this study our model has shown remarkable success in describing the heavy quarkonia data well once the potential parameters are fitted properly for optimum agreement with the experimentally observed data.

We have made the Salpeter binding mass Eq. (26) more general by including the spin-dependent corrections to the potential. Such a formula is able to describe with good accuracy the spectra of all quark-antiquark bound states as long as the mass of the constituent quark is larger than the term $W_{n,l}(r)$ as remarked earlier. Our predictions for the Salpeter mass spectrum of all unobserved and observed charmonium and upsilon systems, to all static potentials, in the flavour-dependent and flavour-independent cases with $\epsilon = 0, 1/2, 1$ are presented in Tables II-III. Further, Table IV shows the predicted Salpeter masses of unobserved $n^{2S+1}D_J$, $n^{2S+1}F_J$ and $n^{2S+1}G_J$ bottomonium levels using a new set of potential parameters. Therefore the calculated bottom masses of all static potentials for the $n^{2S+1}S_J$, $n^{2S+1}P_J$ and $n^{2S+1}D_J$ states are quite fair for the Turin potential as clearly seen from the average relative error range, $\left\langle (M^{Th} - M^{Exp})/M^{Exp} \right\rangle$. We have also predicted the B_c meson spectrum for the three static potentials in Table V using the set of parameters in Table I. In Tables VI and VII we calculated the $c\overline{b}$, $c\overline{c}$ and $b\overline{b}$ using a different set of fitted parameters for Cornell potential. No additive constant is permitted in this manner and the accuracy in producing spectra is pretty good. There is a clear preference in these fits for the Salpeter wave equation with an approximately equal mixture of scalar and vector ($\epsilon = 1/2$) couplings and also vector confinement ($\epsilon = 0$). We found that larger values of ϵ are mildly preferred by fits to the spin-singlet or triplet states while the quality of the $\epsilon = 1$ fit to the spin singlet and triplet data is not as high as that of the $\epsilon = 1/2$ and $\epsilon = 0$ fits, it is

certainly acceptable, with a value of errors. We conclude here that the Lorentz structure of the confining interaction cannot be determined using only $\epsilon = 1/2$. We will henceforth restrict our attention to the case of pure scalar confinement at large distance as expected theoretically. The three potentials seem to be fairly good in fitting all the data..

In the equal scalar and vector couplings, we have found that our fits are very good with level values and accurate to a few MeV. For convenience we compare explicitly the predicted and measured spin splitting energy for different L states. We find that the apparent success is achieved for the predicted $\chi_{b2} - \chi_{b1} = 24$ MeV and $\chi_{b1} - \chi_{b0} = 33$ MeV in the average for the three potentials and are very close to the experimental values 21 MeV and 32 MeV respectively. Furthermore, the predicted $\chi'_{b2} - \chi'_{b1} = 13$ MeV and $\chi'_{b1} - \chi'_{b0} = 16$ MeV in the average for the three potentials which are exactly same as the experimental value 13 MeV and close to 23 MeV, respectively. The predicted hyperfine splitting $\Delta_{\rm hfs}(1{\rm S}) = M(\Upsilon(1{\rm S})) - M(\eta_b(1{\rm S})) = 80^{+6}_{-8}$ MeV, (cf. [33]), $\Delta_{\rm hfs}(2{\rm S}) = M(\Upsilon'(2{\rm S})) - M(\eta'_b(2{\rm S})) = 22^{+3}_{-2}$ MeV, and $\Delta_{\rm hfs}(3{\rm S}) = M(\Upsilon''(3{\rm S})) - M(\eta''_b(3{\rm S})) = 14^{+2}_{-1}$ MeV are nearly close to the theoretically calculated values 62 MeV, 40 MeV, and 15 MeV, respectively, (cf. [33,39]). Further, The hyperfine splitting for the 2S charmonium state is calculated and the predicted number is $\Delta_{\rm hfs}(2{\rm S}) = M(\psi(2{\rm S})) - M(\eta_c(2{\rm S})) = 56^{+4}_{-8}$ MeV for flavour dependent case and 56^{+18}_{-8} MeV for flavour independent case, (cf. [37,39]) . Badalian and Bakker in their recent work calculated and predicted the number as $\Delta_{\rm hfs}(2{\rm S,theory}) = 57 \pm 8$ MeV [37] giving $M(\eta_c(2{\rm S}) = 3630 \pm 8$ MeV.

We observe, in this connection, that splitting approximation can be improved significantly by increasing the quantum number L. It has also been noted that the term contributing most to the Salpeter binding energy is the leading term E_0 as L becomes non zero which improves the convergence of Eq. (25). This is expected since the expansion parameter $1/\overline{k}$ becomes smaller as L becomes larger.

It is found that most results agree moderately well with the experimental data, or with the theoretical predictions wherever experimental data are not available. Our predictions for the lower states are considerably better than the higher states since $\Upsilon(n{\rm S})$, where $n > 4$

lie above the threshold and predominantly decay into mesons containing charm and beauty flavour.[2]

The results obtained are a relevant summary of the effectiveness of the method. It seems likely, that the average relative errors of the fitting results are almost very small for most states. The comparisons with the experimental data show clearly that when we are taking into account the relativistic corrections and also the third order approximation to the binding energy, the coincidence with the experimental data is improved. The deviations from experiment are more considerable at n^1S_0 states for $c\overline{c}$ and $b\overline{b}$ systems. These results show that the last terms in these equations in most cases are not important for the spectrum of meson masses. We do not pretend to display a whole account of the meson spectroscopy. Our purpose is to demonstrate the possibility of applicance of SLNET for determination of meson masses in the context of the SS equation using relativistic kinematics.

We have found that fitting the parameters is extremely essential to enable one to sharpen the analysis.[3] The prescription and the wave equation used are responsible for the deterioration of the fit parameters. The calculation and parameters are also model dependent as remarked in our earlier papers [10,31]. All we can say that the SLNET works for the SS equation well with relativistic kinematics as these parameters are fitted properly and once the mass of the quark is taken large.

¿From a phenomenological point of view, a certain amount of flavour dependence is required. It was pointed out by Miller *et al.* in [14] that no flavour-independent potential could fit both the spin-averaged $c\overline{c}$ and $b\overline{b}$ levels.

It is clear that the coulomb-like parameter A is in accordance with the ideas of asymptotic freedom is expected for the strong gauge-coupling constant of QCD, that is, $\alpha_s(m_c) \neq$

[2]In the upsilon family, Kiselev [48] found $n = 4$.

[3]We have used the available parameter fits in the literature as a test for the present work.

$\alpha_s(m_b) \neq \alpha_s(\mu)$ with $\mu = m_c m_b/(m_c + m_b)$; (cf. e.g., Ref. [48]).[4] Of course a complete analysis would require ingredients that we have not considered here but discussed in the literature.

The calculated values of the pseudoscalar and vector decay constants of the B_c meson in our model using the nonrelativistic formula (65) are displayed in Table VIII. The radial wave function at the origin is being calculated in Table IX. They are compared with the ones calculated using the relativistic and other predictions using the nonrelativistic quark models. The calculated values of these leptonic constants are consistent with the other predictions [22,25,29,38,49,50]. Kiselev *et al.* [49] estimated the pseudoscalar leptonic constant $f_{B_c}^{NR} = 493$ MeV in the potential model. Furthermore, the calculations in the same potential model with the one-loop matching [50] is $f_{B_c}^{1-loop} = 400 \pm 45$ MeV and also for the two-loop calculations [50] is $f_{B_c}^{2-loop} = 395 \pm 15$ MeV.

We may expect to improve our results by adding extra spin-independent terms comparable to the ones induced by the original scalar confinement potential.

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[4] For a pretty good fit to the $c\overline{c}$, $b\overline{b}$ and $c\overline{b}$ quarkonium spectra, the QCD coupling constant $\alpha_s(\mu^2)$ must be dependent on the quark-flavour. Kiselev [48] took $\alpha_s(m_b)/\alpha_s(m_c) \simeq 3/4$.

APPENDIX A: THE SPIN-CORRECTION TERMS:

For parastates ($S = 0$) case we have:

$$J = L \tag{A1}$$

For triplet ($S = 1$) case we have the following:

$$J = \begin{cases} L-1,\ \mathbf{S}\cdot\mathbf{L} = -(L+1) \\ L,\ \mathbf{S}\cdot\mathbf{L} = -1 \\ L+1,\ \mathbf{S}\cdot\mathbf{L} = L \end{cases} \tag{A2}$$

For bound-state constituents of spin $S_1 = S_2 = 1/2$, the independent operators $\mathbf{S}_1 \cdot \mathbf{S}_2$,:$(\mathbf{S}_1 \pm \mathbf{S}_2)\cdot\mathbf{L}$ and T :

$$\langle \mathbf{S}_1 \cdot \mathbf{S}_2 \rangle = \begin{cases} -3/4 \text{ for spin singlets, } S = 0, \\ +1/4 \text{ for spin triplets, } S = 1. \end{cases} \tag{A3}$$

$$\langle \mathbf{S}\cdot\mathbf{L} \rangle = \begin{cases} 0,\ \text{for spin singlets } S = 0, \\ \frac{1}{2}\left[J(J+1) - L(L+1) - 2\right],\ \text{for spin triplets } S = 1. \end{cases} \tag{A4}$$

$$(\mathbf{S}_1\cdot\hat{\mathbf{r}}\mathbf{S}_2\cdot\hat{\mathbf{r}})u_J(r)Y_{J,0,J}(\hat{\mathbf{r}}) = -\frac{1}{4}u_J(r)Y_{J,0,J}(\hat{\mathbf{r}}), \tag{A5}$$

$$(\mathbf{S}_1\cdot\mathbf{S}_2)Y^m_{J,S,L}(\hat{\mathbf{r}}) = \frac{1}{2}\left[S(S+1) - S_1(S_1+1) - S_2(S_2+1)\right]Y^m_{J,S,L}(\hat{\mathbf{r}}), \tag{A6}$$

$$(\mathbf{S}_1+\mathbf{S}_2)\cdot\mathbf{L}Y^m_{J,S,L}(\hat{\mathbf{r}}) = \frac{1}{2}\left[J(J+1) - L(L+1) - S(S+1)\right]Y^m_{J,S,L}(\hat{\mathbf{r}}), \tag{A7}$$

$$(\mathbf{S}_1-\mathbf{S}_2)\cdot\mathbf{L}Y^m_{J,S,L}(\hat{\mathbf{r}}) = \sqrt{\frac{1}{10}\left[2L+3)(2L-1)\right]}\delta_{J,L}\left(\delta_{S,0}Y^m_{J,1,L}(\hat{\mathbf{r}}) + \delta_{S,1}Y^m_{J,0,L}(\hat{\mathbf{r}})\right), \tag{A8}$$

$$TY^m_{J,1,L}(\hat{\mathbf{r}}) = \frac{1}{4}\delta_{J,L}Y^m_{J,1,L}(\hat{\mathbf{r}}) - \frac{1}{4(2L-1)}\delta_{J,L-1}Y^m_{J,1,L}(\hat{\mathbf{r}}) + \frac{1}{4(2L+3)}\delta_{J,L+1}Y^m_{J,1,L}(\hat{\mathbf{r}})$$

$$-\frac{\sqrt{(L+1)(L+2)}}{2(2L+3)}\delta_{J,L+1}Y^m_{J,1,L+2}(\hat{\mathbf{r}}) - \frac{\sqrt{L(L-1)}}{2(2L-1)}\delta_{J,L-1}Y^m_{J,1,L-2}(\hat{\mathbf{r}}). \tag{A9}$$

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TABLES

TABLE I. Fitted parameters of the class of static central potentials.

	Cornell	Song-Lin	Turin
$c\overline{c}$			
m_c	1.84 GeV	1.80 GeV	1.82 GeV
A	0.509	0.498 $GeV^{1/2}$	0.547 $GeV^{1/4}$
κ	0.166 GeV^2	0.694 $GeV^{3/2}$	0.309 $GeV^{7/4}$
V_0	-0.799 GeV	-1.314 GeV	-0.898 GeV
$b\overline{b}$			
m_b	5.17 GeV	5.20 GeV	5.18 GeV
A	0.443	0.762 $GeV^{1/2}$	0.576 $GeV^{1/4}$
κ	0.184 GeV^2	0.576 $GeV^{3/2}$	0.304 $GeV^{7/4}$
V_0	-0.805 GeV	-0.991 GeV	-0.845 GeV
Combined			
m_c	1.79 GeV	1.82 GeV	1.80 GeV
m_b	5.17 GeV	5.20 GeV	5.18 GeV
A	0.457	0.722 $GeV^{1/2}$	0.578 $GeV^{1/4}$
κ	0.182 GeV^2	0.589 $GeV^{3/2}$	0.304 $GeV^{7/4}$
V_0	-0.790 GeV	-1.029 GeV	-0.838 GeV

TABLE II. Salpeter masses of the unobserved $c\overline{c}$ and $b\overline{b}$ for the flavor-dependent and flavor-independent cases, respectively for the Cornell, Song-Lin and Turin potentials

State[a]	[45]	Cornell[b]	Song-Lin[b]	Turin[b]	Cornell[c]	Song-Lin[c]	Turin[c]
$\eta_c(2S)$[d]	3589	3604	3595	3593	3602	3574	3585
$(1^1D_2)c\overline{c}$	3826	3755	3756	3756	3748	3756	3754
$(1^3D_1)c\overline{c}$	3814	3708	3708	3705	3699	3703	3701
$\eta_b(1S)$	9314	9398	9407	9395	9380	9433	9399
$\eta_b(2S)$	9931	10000	10000	9997	10000	10014	10003
$\eta_b(3S)$	10288	10342	10329	10332	10343	10341	10338
$\eta_b(4S)$	10577	10612	10570	10586	10613	10581	10592
$(1^1P_1)b\overline{b}$	9906	9905	9890	9897	9908	9901	9902
$(2^1P_1)b\overline{b}$	10254	10246	10246	10245	10249	10256	10251
$(1^1D_2)b\overline{b}$	10155	10142	10149	10146	10145	10157	10152
$(2^1D_2)b\overline{b}$	10450	10424	10422	10423	10427	10431	10429
$\chi_{b0}(3P)$	10519	10512	10491	10498	10513	10501	10504
$\chi_{b1}(3P)$	10542	10520	10500	10508	10522	10510	10514
$\chi_{b2}(3P)$	10561	10528	10508	10516	10530	10518	10522
$(1^3D_1)b\overline{b}$	10147	10128	10132	10131	10131	10142	10137
$(1^3D_2)b\overline{b}$	10153	10140	10147	10145	10144	10156	10151
$(1^3D_3)b\overline{b}$	10158	10152	10161	10158	10156	10170	10164
$(2^3D_1)b\overline{b}$	10442	10414	10411	10412	10417	10420	10418
$(2^3D_2)b\overline{b}$	10448	10423	10421	10422	10426	10430	10428
$(2^3D_3)b\overline{b}$	10453	10433	10431	10432	10436	10440	10438

[a]States with equal mixtures of scalar and vector couplings.

[b]Here's the flavour-dependent case.

[c]Here's the flavour-independent case.

[d]Recently observed by Belle Collaboration [34,35].

TABLE III. Salpeter $c\overline{c}$ and $b\overline{b}$ quarkonium masses of the observed states (in MeV) for the Cornell, Song-Lin and Turin potentials

State[a]	$M^{(\exp)}$[36]	[45]	Cornell[b]	Song-Lin[b]	Turin[b]	Cornell[c]	Song-Lin[c]	Turin[c]
$\eta_c(1S)$	2980	2967	2981	2965	2950	2988	2983[d]	2999[d]
$\psi(1S)$	3097	3126	3104	3108	3111	3092	3096[d]	3087[d]
$\psi(2S)$	3686	3700	3652	3655	3652	3650	3640	3639
$\chi_{c0}(1P)$	3415	3427	3316	3374	3335	3322	3340	3309
$\chi_{c1}(1P)$	3511	3497	3471	3461	3464	3456	3463	3459
$\chi_{c2}(1P)$	3556	3543	3535	3521	3530	3521	3532	3529
$h_c(1P)$	3526	3510	3488	3468	3478	3471	3475	3474
$\Upsilon(1S)$	9460	9511	9474	9499	9488	9466	9505[d]	9480[d]
$\Upsilon(2S)$	10023	10004	10019	10034	10024	10020	10039[d]	10025[d]
$\Upsilon(3S)$	10355	10350	10354	10350	10349	10356	10357	10352
$\Upsilon(4S)$	10580	10642	10622	10585	10599	10624	10593	10603
$\chi_{b0}(1P)$	9860	9854	9863	9854	9855	9863	9868	9860
$\chi_{b1}(1P)$	9892	9878	9898	9885	9890	9900	9885	9896
$\chi_{b2}(1P)$	9913	9896	9921	9910	9915	9924	9920	9921
$\chi_{b0}(2P)$	10232	10213	10228	10228	10226	10230	10239	10232
$\chi_{b1}(2P)$	10255	10236	10243	10244	10242	10245	10254	10248
$\chi_{b2}(2P)$	10268	10254	10255	10257	10256	10258	10266	10262

[a]States with equal mixtures of scalar and vector couplings.

[b]Here's the flavour-dependent case.

[c]Here's the flavour-independent case.

[d]The hyperfine splitting term is not considered here.

TABLE IV. Salpeter $b\overline{b}$ quarkonium masses (in MeV) for the Cornell, Song-Lin and Turin potentials

State[a]	[45]	Cornell	Error %	Song-Lin	Error %	Turin	Error %
1^1S_0[b]	9398	9342	0.60	9344	0.57	9343	0.58
1^3S_1	9460	9438	0.23	9464	0.04	9457	0.03
1^3P_2	9913	9912	0.01	9900	0.13	9907	0.06
1^3P_1	9891	9886	0.05	9872	0.19	9879	0.12
1^3P_0	9870	9846	0.24	9834	0.36	9838	0.32
1^1P_1	9908	9894	0.14	9878	0.30	9887	0.21
1^3D_3	-	10149		10162		10157	
1^3F_4	-	10335		10356		10348	
1^3G_5	-	10496		10514		10507	
2^1S_0	9983	9986	0.03	9983	0.0	9984	0.01
2^3S_1	10023	10008	0.15	10024	0.01	10015	0.08
3^3S_1	10355	10351	0.04	10349	0.06	10347	0.08
4^3S_1	10580	10623	0.41	10588	0.07	10601	0.20
Fits[c]	A	0.470		0.869 $GeV^{1/2}$		0.620 $GeV^{1/4}$	
	κ	0.186 GeV^2		0.558 $GeV^{3/2}$		0.304 $GeV^{7/4}$	
	V_0	-0.802 GeV		-0.893 GeV		-0.823 GeV	
	m_b	5.17 GeV		5.20 GeV		5.18 GeV	

[a]States with equal mixtures of scalar and vector couplings.

[b]Recently searched by ALEPH Collaboration [33].

[c]Here we cite Ref. [10] for the parameter fits.

TABLE V. B_c meson mass spectrum (in MeV) for the Cornell, Song-Lin and Turin potentials, with a flavour-independent case.

State	$\epsilon=1$	$\epsilon=1/2$	$\epsilon=0$	$\epsilon=1$	$\epsilon=1/2$	$\epsilon=0$	$\epsilon=1$	$\epsilon=1/2$	$\epsilon=0$	[45]	[50]
1^1S_0	6296.9	6284.9	6272.7	6287.2	6269.3	6250.2	6278.6	6263.3	6247.3	6270	6253
2^1S_0	6852.5	66846.7	6840.8	6855.0	6848.5	6841.9	6850.0	6843.9	6837.7	6835	6867
3^1S_0	7220.1	7215.9	7211.7	7190.4	7186.7	7182.8	7200.9	7196.9	7193.0	7193	
4^1S_0	7520.3	7516.9	7513.5	7441.1	7438.5	7435.9	7474.4	7471.4	7468.5		
1^1P_1	6724.8	6719.9	6714.9	6726.6	6721.8	6717.0	6725.9	6721.1	6716.2	6734	6717
2^1P_1	7100.4	7096.6	7092.7	7091.7	7088.5	7085.3	7095.0	7091.5	7088.1	7126	7113
1^1D_2	6979.9	6976.5	6973.0	6992.9	6990.3	6987.7	6988.3	6985.3	6982.3	7077	7001
2^1D_2	7298.9	7296.0	7293.0	7277.4	7275.4	7273.4	7287.4	7285.0	7282.5		
1^3P_1	6717.5	6706.5	6695.4	6724.9	6710.9	6696.5	6719.7	6707.5	6695.0	6749	6729
2^3P_1	7098.4	7090.1	7081.8	7091.7	7082.7	7073.4	7093.1	7084.6	7076.0	7145	7124
1^3D_2	6981.4	6974.0	6966.6	6994.5	6987.2	6979.8	6989.4	6982.2	6974.9	7079	7016
2^3D_2	7300.4	7294.0	7287.7	7278.7	7273.1	7267.4	7288.4	7282.6	7276.6		
1^3S_1	6296.9	6300.8	6304.8	6322.7	6327.6	6332.4	6310.1	6314.4	6318.6	6332	6317
2^3S_1	6852.5	6854.4	6856.4	6862.4	6864.5	6866.6	6854.4	6856.3	6858.3	6881	6902
3^3S_1	7220.1	7221.5	7222.9	7193.5	7194.7	7196.0	7202.5	7203.8	7205.1	7235	
4^3S_1	7520.3	7521.4	7522.6	7442.8	7443.6	7444.5	7475.2	7476.2	7477.2		
1^3P_2	6735.9	6748.5	6762.0	6739.2	6755.1	6771.8	6747.0	6753.6	6768.0	6762	6743
2^3P_2	7104.7	7115.6	7126.4	7097.9	7109.2	7120.2	7101.4	7112.2	7122.8	7156	7134
1^3D_3	6979.4	6998.1	7016.3	6996.1	7013.6	7030.9	6990.4	7008.3	7025.3	7081	7007
2^3D_3	7298.3	7314.4	7331.1	7279.4	7293.3	7307.5	7288.9	7303.6	7318.8		
1^3P_0	6673.1	6639.2	6602.9	6695.8	6652.0	6602.0	6676.7	6636.7	6592.3	6699	6683
2^3P_0	7082.4	7060.7	7038.6	7078.0	7052.4.	7025.2	7075.9	7052.5	7028.2	7091	7088
1^3D_1	6977.4	6949.6	6921.2	6988.9	6960.1	6929.5	6983.2	6955.1	6926.0	7072	7008
2^3D_1	7297.8	7274.1	7250.2	7274.8	7252.9	7230.0	7284.1	7261.6	7238.5		

TABLE VI. Flavor-independent $c\overline{b}$ mass spectrum (in MeV) using the Cornell potential with scalar ($\epsilon = 1$), an equal mixture of scalar-vector ($\epsilon = 1/2$) and vector ($\epsilon = 0$) interactions.

State	$\epsilon = 1$[a]	$\epsilon = 1/2$[b]	$\epsilon = 0$[c]	State	$\epsilon = 1$	$\epsilon = 1/2$	$\epsilon = 0$	Parameters [25]
1^1S_0	6302.6	6284.5	6265.9	1^3S_1	6302.6	6308.5	6314.4	$A = 0.437$
2^1S_0	6884.6	6875.2	6865.6	2^3S_1	6884.6	6887.7	6890.8	$\kappa = 0.203\ GeV^2$
3^1S_0	7285.1	7278.2	7271.2	3^3S_1	7285.1	7287.4	7289.7	$V_0 = 0$[d]
4^1S_0	7614.6	7608.9	7603.1	4^3S_1	7614.6	7616.5	7618.4	$m_c = 1.321\ GeV$
1^1P_1	6740.7	6732.9	6725.0	1^3P_2	6746.9	6773.4	6796.6	$m_b = 4.731\ GeV$
2^1P_1	7152.2	7146.0	7139.7	2^3P_2	7155.6	7175.1	7196.3	
1^1D_2	7019.4	7013.8	7008.2	1^3D_3	7012.2	7046.2	7078.8	
2^1D_2	7371.0	7366.1	7361.2	2^3D_3	7366.5	7394.9	7411.0	
1^3P_1	6734.5	6714.3	6693.8	1^3P_0	6677.3	6608.7	6525.4	
2^3P_1	7151.8	7136.3	7120.5	2^3P_0	7132.4	7091.6	7049.0	
1^3D_2	7024.1	7010.5	6996.8	1^3D_1	7024.6	6973.8	6921.0	
2^3D_2	7375.2	7363.2	7351.2	2^3D_1	7376.1	7332.2	7286.9	

[a]Here $\epsilon = 1$ is the scalar confinement interaction.

[b]Here $\epsilon = 1/2$ is the equal mixture of scalar-vector confinement interaction.

[c]Here $\epsilon = 0$ is the vector confinement interaction.

[d]It's a shifting parameter (in MeV) to be set to fix the the position of $M(B_c)$ and $M(B_c^*)$.

TABLE VII. $c\overline{c}$ and $b\overline{b}$ mass spectra (in MeV) using the Cornell potential.

State	Meson[a]	Theory [45]	$\epsilon = 1$	$\epsilon = 1/2$	$\epsilon = 0$	Meson	Theory [45]	$\epsilon = 1$	$\epsilon = 1/2$	$\epsilon = 0$
1^1S_0	η_c	2979	3068.1	3020.8	2970.7	η_b	9400	9448.3	9441.3	9434.2
2^1S_0	η_c'	3588	3686.7	3659.6	3631.9	η_b'	9993	10021.5	10018.3	10015.0
3^1S_0	η_c''	3991	4135.4	4115.0	4094.2	η_b''	10328	10380.4	10378.1	10375.9
4^1S_0			4510.7	4493.8	4476.6			10669.8	10668.0	10666.2
1^1P_1	h_c	3526	3505.7	3482.7	3459.3	h_b	9901	9901.3	9898.4	9895.5
2^1P_1	h_c'	3945	3974.1	3955.5	3936.7	h_b'	10261	10266.0	10263.9	10261.7
1^1D_2		3811	3812.7	3796.0	3779.1		10158	10150.5	10148.5	10146.5
2^1D_2			4218.2	4203.6	4188.8			10456.4	10454.8	10453.1
1^3P_1	χ_{c1}	3510	3500.6	3463.7	3425.7	χ_{b1}	9892	9894.9	9890.5	9886.0
2^3P_1	χ_{c1}'	3929	3975.5	3946.9	3917.8	χ_{b1}'	10255	10263.7	10260.5	10257.2
1^3D_2		3813	3820.8	3795.6	3770.0		10158	10150.1	10147.1	10144.1
2^3D_2			4225.4	4203.2	4180.9			10456.3	10453.8	10451.3
1^3S_1	J/ψ	3096	3068.1	3083.3	3098.2	Υ	9460	9448.3	9450.6	9452.9
2^3S_1	ψ'	3686	3686.7	3695.6	3704.4	Υ'	10023	10021.5	10022.6	10023.7
3^3S_1	ψ''	4088	4135.4	4142.1	4148.8	Υ''	10355	10380.4	10381.2	10381.9
4^3S_1	ψ'''		4510.7	4516.3	4521.9	Υ'''		10669.8	10670.4	10671.0
1^3P_2	χ_{c2}	3556	3517.1	3566.0	3613.5	χ_{b2}	9913	9910.3	9916.7	9923.0
2^3P_2	χ_{c2}'	3972	3978.8	4019.3	4059.0	χ_{b2}'	10268	10269.9	10274.7	10279.5
1^3D_3		3815	3802.8	3871.4	3937.8		10162	10153.1	10161.5	10169.8
2^3D_3			4210.1	4270.6	4329.5			10458.2	10465.1	10472.1
1^3P_0	χ_{c0}	3424	3369.4	3106.0	-	χ_{b0}	9863	9864.0	9850.7	9837.2
2^3P_0	χ_{c0}'	3854	3937.0	3849.1	3752.9	χ_{b0}'	10234	10251.9	10243.0	10234.0
1^3D_1		3798	3817.8	3711.0	3594.8		10153	10144.3	10132.4	10120.4
2^3D_1			4224.0	4132.6	4035.6			10452.3	10442.5	10432.7

[a]Same parameter fits as in Table VI.

TABLE VIII. Pseudoscalar and vector decay constants ($f_P = f_{B_c},\ f_V = f_{B_c^*}$) of the B_c meson (in MeV).

Constants	SLNET[a]	rel [45]	NR [45]	[23]	[29]	[25]	[48]
f_{B_c}	378-513	433	562	479-687	460±60	517	420±13
$f_{B_c^*}$	347-482	503	562	479-687	460±60	517	

[a]We used different potential models with ($\epsilon = 0$).

TABLE IX. The radial wave function at the origin (in GeV^3) calculated in our model and by the other authors.

Level	SLNET[a]	Martin	EQ[23]	F[25]	[23][b]
$\left\|R_{B_c}(0)\right\|^2$	0.94-1.73	1.716	1.638	1.81	1.508-3.102
$\left\|R_{B_c^*}(0)\right\|^2$	0.73-1.52				

[a]We used different potential models with ($\epsilon = 0$).

[b]For the 1S level.